Solitary Wave Solutions of High Order Scalar Fields and Coupled Scalar Fields

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An arbitrary Klein-Gordon field with a quite general constrained condition (which contains an arbitrary function) can be used as an auxilialy field such that some special types of solutions of high order scalar fields can be obtain by solving an ordinary differential equation (ODE). For a special type of constraint, the general solution of the ODE can be obtained by twice integrating. The solitary wave solutions of the ϕ^5 model are treated in an alternative simple way. The obtained solutions of the ϕ^5 model can be changed to those of the ϕ^8 field and coupled scalar fields.

1. Introduction

The Lagrangian density of a generalized nonlinear Klein-Gordon (NKG) field $\phi \equiv \phi(x_1, x_2, ...x_D, t)$ in $(D+1)$-dimensions has the form

$$\mathcal{L}[\phi, \partial_\mu \phi] = \partial_\mu \phi \partial^\mu \phi - V(\phi), \tag{1}$$

and the corresponding equation of wave motion reads

$$\Box\phi \equiv \sum_{i=1}^{D} \phi_{x_i x_i} - \phi_{tt} = F(\phi), \ \left(F(\phi) \equiv \frac{\mathrm{d}V}{\mathrm{d}\phi}\right). \tag{2}$$

For not very large ϕ, the potential $V \equiv V(\phi)$ in (1) can be replaced by a polynomial function of ϕ:

$$V = \sum_{n=0}^{N} \frac{1}{n} v_n \phi^n, \ \left(v_n = \frac{1}{(n-1)!} \frac{\mathrm{d}^n v}{\mathrm{d}\phi^n}\Big|_{\phi=0}\right), \tag{3}$$

and then the related wave motion equation becomes

$$\Box\phi = \sum_{n=1}^{N} v_n \phi^{n-1}. \tag{4}$$

Some well known NKG models are just special cases of (3) (or (4)), say, the ϕ^4 model corresponding to $N = 4, \ v_1 = v_3 = 0$ [1], the ϕ^6 model to $N = 6, \ v_1 = v_3 = v_5 = 0$ [2], and the $\phi^3 + \phi^4$ (or Friedberg-Lee (FL)) model [3] is related to $N = 4, \ v_1 = 0$. In [4], the authors discuss the interesting properties of the self-exited soliton motion by using a ϕ^{10} ($N = 10, \ v_{2i+1} = 0, \ (i = 0...4)$) model. In [5] the generalized ϕ^{2m} model ($N = 2m, \ v_{2i+1} = 0, \ (i = 0, 1, ..., m-1)$) is used to study the effective kink-kink interaction mediated by phonon exchange. However, to our knowledge there are no known exact solutions of (4) for $N = 5$ and $N > 6$. In Sects. 2 and 3 of this paper we study the exact solitary wave solutions of (4) for $N = 5$. In Sect. 2, using an arbitrary scalar field with a quite general constraint as a basic equation system, we solve the $(D+1)$-dimensional ϕ^5 model by a second order ordinary differential equation (ODE). For a special constraint, the general solution of the ODE can be expressed simply by an integration. The solitary wave solutions of the ϕ^5 model are discussd in an alternative way in Section 3. In Sect. 4, we give special solutions of the ϕ^8 model ($v_1 = v_3 = v_5 = v_7 = 0$ in (4)) by means of the ϕ^5 model. In Sect. 5, we discuss some special solutions of a coupled scalar field model by means of the ϕ^5 model. The last section is a short summary.

2. Base Equation Approach for the ϕ^5 Model

For the ϕ^5 model, the motion eq. (4) becomes

$$\Box\phi = v_2\phi + v_3\phi^2 + v_4\phi^3 + v_5\phi^4, \tag{5}$$

where we have set $v_1 = 0$ without loss of generality because we can make transformation $\phi \to \phi + c$ by

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selecting the constant c appropriately. In order to get some interesting special solutions of (5), we introduce a basic equation system [4, 5]

$$\Box\psi = A(\psi), \tag{6}$$

$$\partial_\mu\psi\partial^\mu\psi \equiv \sum_{i=1}^{D}(\psi_{x_i})^2 - (\psi_t)^2 = B(\psi), \tag{7}$$

where $A \equiv A(\psi)$ and $B \equiv B(\psi)$ are arbitrary functions of ψ. Now we suppose that ϕ is only a function of ψ. In other words, the space-time dependence of $\phi = \phi(\psi)$ results from the auxiliary field ψ, which is given by (6) and (7) with two arbitrary functions A and B. Using the basic eqs. (6) and (7) and the above ansatz, (5) becomes an ODE:

$$A\phi_\psi + B\phi_{\psi\psi} = v_2\phi + v_3\phi^2 + v_4\phi^3 + v_5\phi^4. \tag{8}$$

Furthermore, after introducing

$$C(\psi) = \exp\left\{2\int^{\psi}\frac{1}{B(\psi')}\left[A(\psi') - \frac{1}{2}\frac{\mathrm{d}B(\psi')}{\mathrm{d}\psi'}\right]\mathrm{d}\psi'\right\}, \tag{9}$$

the ODE is changed to

$$\phi_{\xi\xi} = (v_2\phi + v_3\phi^2 + v_4\phi^3 + v_5\phi^4)C(\psi(\xi)), \tag{10}$$

where ξ and ψ are related by

$$\xi = \int^{\psi}\frac{d\psi'}{\sqrt{B(\psi')C(\psi')}}. \tag{11}$$

Generally, to solve the ODE (10) is still quite difficult. However, if we select

$$A = \frac{1}{2}\frac{\mathrm{d}B}{\mathrm{d}\psi}, \tag{12}$$

then $C(\psi) = 1$, and (10) is reduced to

$$\phi_{\xi\xi} = v_2\phi + v_3\phi^2 + v_4\phi^3 + v_5\phi^4, \ \left(\xi = \int^{\psi}\frac{d\psi'}{\sqrt{B(\psi')}}\right) \tag{13}$$

with the general solution

$$\xi = \pm\int^{\phi}\frac{d\phi'}{\sqrt{C_1 + v_2\phi'^2 + \frac{2}{3}v_3\phi'^3 + \frac{1}{2}v_4\phi'^4 + \frac{2}{5}v_5\phi'^5}} + C_2, \tag{14}$$

where C_1 and C_2 are two arbitrary integral constants.

Now the remaining key problems are how to solve the basic eqs. (6) and (7) and to finish the integration (14). For the first problem, the concrete solutions of (6) are dependent on the selection of the arbitrary function B. If we selected the function B as the potential of the known NKG fields, say, sine-Gordon (sG), ϕ^4 or ϕ^6 models, various types of special solutions have been given[6]. The simplest nontrivial selection of (6) and (7) reads

$$\Box\psi = \sigma^2\psi, \tag{15}$$

$$\partial_\mu\psi\partial^\mu\psi = \sigma^2\psi^2. \tag{16}$$

In this simple case we have

$$\xi = \frac{\ln\psi}{\sigma}. \tag{17}$$

The simplest solution of (15) and (16) has the form

$$\psi = \left[\sum_{\gamma=1}^{N} c_\gamma \exp\left(\frac{\sigma}{\sigma_1}\theta_\gamma\right)\right]^{\sigma_1}, \tag{18}$$

where $\sigma,\ \sigma_1$ and $c_\gamma, \gamma = 1, 2, \ldots, N$ are arbitrary constants and

$$\theta_\gamma = \sum_{i=1}^{D} P_\gamma^i x_i - \omega_\gamma t, \tag{19}$$

while P_γ^i and ω_γ satisfy the conditions

$$\sum_{i=1}^{D} P_\gamma^i P_{\gamma'}^i - \omega_\gamma\omega_{\gamma'} = 1, \ \gamma, \gamma' = 1, 2, \ldots, N. \tag{20}$$

3. Solitary Wave Solutions of the ϕ^5 Model

Generally, the integration of (14) can not be expressed by simple functions. For the solitary wave solutions with some special parameters (v_i) and integration constant (C_1) we can take an alternative way proposed, in [7] to express (14) by some known functions.

Equation (14) is equivalent to

$$\phi_\xi^2 = C_1 + v_2\phi^2 + \frac{2}{3}v_3\phi^3 + \frac{1}{2}v_4\phi^4 + \frac{2}{5}v_5\phi^5. \tag{21}$$

To find the solitary wave solutions of (21), we take a truncated series expansion of ϕ around an extended singular manifold $\chi = 0$, while χ is determined by [7]

$$\chi_\xi = \sum_{k=0}^{K} a_k \chi^k, \tag{22}$$

where $a_k,\ k = 0, 1, \ldots K$ are arbitrary functions of ξ. If the integer K is fixed, the possible expansions of ϕ read

$$\phi = \sum_{p=0}^{\frac{2}{3}(K-1)} b_p \chi^p \tag{23}$$

with $b_p,\ p = 0, 1, \ldots \frac{2}{3}(K-1)$ being functions of ξ. For simplicity, we take

$$K = 4. \tag{24}$$

Substituting (23) with (22) and (24) into (21) and putting to zero the coefficients of $\chi^k, k = 0, \ldots$, we have eleven complicated equations of the functions $a_0, a_1, a_2, a_3, a_4, b_0, b_1, b_2$ and the constant C_1. If the parameters v_2, v_3, v_4 and v_5 satisfy the conditions

$$\begin{aligned} W \equiv\ & 11232 v_5^2 v_4^2 v_3^2 - 5400 v_5 v_4^4 v_3 + 9504 v_5^2 v_2 v_4^3 \\ & - 35712 v_5^3 v_2 v_4 v_3 + 31104 v_5^4 v_2^2 \\ & + 675 v_4^6 - 2048 v_5^3 v_3^3 \neq 0 \end{aligned} \tag{25}$$

and

$$\begin{aligned} & (640 v_5 v_3^3 - 150 v_4^2 v_3^2 + 675 v_4^3 v_2 - 3240 v_3 v_5 v_4 v_2 \\ & + 5832 v_5^2 v_2^2)(-128 v_5^2 v_3^3 - 45 v_4^4 v_3 + 222 v_3^2 v_5 v_4^2 \\ & + 27 v_5 v_4^3 v_2 - 504 v_3 v_5^2 v_4 v_2 + 216 v_5^3 v_2^2) = 0, \end{aligned} \tag{26}$$

we have an unique solution for a_i, b_i and C_1 being constants:

$$a_4 = \frac{1}{10}\sqrt{10 v_5 b_2}\, b_2, \quad a_3 = \frac{1}{5} b_1 \sqrt{10 v_5 b_2}, \tag{27}$$

$$a_2 = \frac{1}{8\sqrt{10 v_5 b_2}}(7 v_5 b_1^2 + 20 v_5 b_0 b_2 + 5 v_4 b_2), \tag{28}$$

$$a_1 = \frac{b_1}{8\sqrt{10 v_5 b_2^3}}(-v_5 b_1^2 + 20 v_5 b_0 b_2 + 5 v_4 b_2), \tag{29}$$

$$\begin{aligned} a_0 = \frac{1}{384\sqrt{10 v_5^3 b_2^5}} \Big(& 9 v_5^2 b_1^4 - 120 v_5^2 b_0 b_1^2 b_2 \\ & - 30 v_4 b_1^2 b_2 v_5 + 720 v_5^2 b_0^2 b_2^2 \\ & + 360 v_4 b_0 b_2^2 v_5 - 75 v_4^2 b_2^2 + 320 v_3 b_2^2 v_5 \Big), \end{aligned} \tag{30}$$

$$C_1 = -\frac{2}{3} v_3 b_0^3 - \frac{2}{5} v_5 b_0^5 - v_2 b_0^2 + b_1^2 a_0^2 - \frac{1}{2} v_4 b_0^4, \tag{31}$$

$$\begin{aligned} b_0 = -\frac{1}{4 b_2 v_5 W} \Big(& 51840 v_5^2 b_2 v_2 v_4^4 - 9504 v_5^3 v_2 v_4^3 b_1^2 \\ & + 35712 v_5^4 v_2 v_4 b_1^2 v_3 + 98304 v_5^4 b_2 v_2 v_3^2 \\ & - 31104 v_5^5 v_2^2 b_1^2 - 233856 v_5^3 b_2 v_2 v_4^2 v_3 \\ & + 196992 v_5^4 b_2 v_2^2 v_4 + 2048 v_3^3 v_5^4 b_1^2 \\ & + 5400 v_5^2 v_4^4 v_3 b_1^2 - 11232 v_5^3 v_4^2 v_3^2 b_1^2 \\ & - 51200 v_5^3 b_2 v_4 v_3^3 - 30600 v_5 b_2 v_4^5 v_3 \\ & + 81120 v_5^2 b_2 v_4^3 v_3^2 + 3375 b_2 v_4^7 - 675 v_4^6 b_1^2 v_5 \Big) \end{aligned} \tag{32}$$

while b_1 and b_2 remain free.

Now the remaining problem is to solve the ODE (22) with (27) - (30) and (32). According to the different relations among the model parameters v_2, v_3, v_4, v_5 and the constants b_1 and b_2, there may be nine types of possible χ solutions:

1.) If the quartic equation

$$\sum_{k=0}^{4} a_k \chi^k = 0, \tag{33}$$

a_i being given by (27) - (30), possesses four non-degenerate real roots, say $\{c_1, c_2, c_3, c_4\}$, the general solution of (22) reads

$$\sum_{k=0}^{4} \frac{\ln(\chi - c_k)}{\prod_{j\neq k}(c_j - c_k)} - a_4(\xi - \xi_0) = 0, \tag{34}$$

where a_i are related to c_i by

$$a_0/a_4 = c_1 c_2 c_3 c_4, \tag{35}$$

$$a_1/a_4 = -c_1 c_2 c_4 - c_1 c_2 c_3 - c_2 c_3 c_4 - c_1 c_3 c_4, \tag{36}$$

$$a_2/a_4 = c_1c_2 + c_2c_4 + c_2c_3 + c_1c_4 + c_1c_3 + c_3c_4, \quad (37)$$

$$a_3/a_4 = -c_1 - c_2 - c_3 - c_4. \quad (38)$$

2.) If (33) possesses four real roots and two roots are degenerate, say, $c_4 = c_1$, then the solution of (22) has the form

$$\frac{\ln(\chi - c_2)}{(c_2 - c_1)^2(c_3 - c_2)} - \frac{(c_2 - 2c_1 + c_3)\ln(\chi - c_1)}{(-c_1 + c_2)^2(-c_1 + c_3)^2} + \frac{\ln(\chi - c_3)}{(c_2 - c_3)(c_3 - c_1)^2} + \frac{1}{(c_2 - c_1)(c_3 - c_1)(\chi - c_1)} + a_4(\xi - \xi_0) = 0. \quad (39)$$

3.) If three of the four roots of (33) are equal, $c_3 = c_4 = c_1$, the general solution of (22) becomes

$$\frac{\ln(\chi - c_1)}{(-c_1 + c_2)^3} + \frac{1}{(c_2 - c_1)^2(c_1 - \chi)} - \frac{1}{2(c_2 - c_1)(\chi - c_1)^2} - \frac{\ln(\chi - c_2)}{(c_2 - c_1)^3} + a_4(\xi - \xi_0) = 0. \quad (40)$$

4.) If all four real roots of (33) are degenerate, the function χ becomes

$$\chi = -\frac{1}{(3a_4(\xi - \xi_0))^{1/3}} + c_1. \quad (41)$$

5.) If there are two sets of degenerate roots of (33) say, $c_4 = c_1,\ c_3 = c_2$, the solution of (22) has the form

$$\frac{2}{c_1 - c_2}\ln\frac{\chi - c_1}{\chi - c_2} + \frac{1}{\chi - c_1} + \frac{1}{\chi - c_2} + a_4(c_2 - c_1)^2(\xi - \xi_0) = 0. \quad (42)$$

6.) If two of the roots are conjugate complex and the other two are nondegenerate real, we have

$$\frac{d_1c_3 + 2c_3c_4 + 2d_2 + d_1^2 + d_1c_4}{(d_1c_3 + c_3^2 - d_2)(d_2 - d_1c_4 - c_4^2)\sqrt{-4d_2 - d_1^2}}\arctan\left(\frac{2\chi + d_1}{\sqrt{-4d_2 - d_1^2}}\right) - \frac{\ln(\chi - c_4)}{(c_3 - c_4)(d_2 - d_1c_4 - c_4^2)}$$
$$+ \frac{\ln(\chi - c_3)}{(c_3 - c_4)(d_2 - d_1c_3 - c_3^2)} - \frac{(c_3 + d_1 + c_4)\ln(\chi^2 + d_1\chi - d_2)}{(d_2 - d_1c_4 - c_4^2)(-d_1c_3 - c_3^2 + d_2)} + a_4(\xi - \xi_0) = 0. \quad (43)$$

where

$$d_1^2 + 4d_2 < 0, \quad (44)$$

and the a_i are linked with $\{d_1, d_2\}$ and $\{c_3, c_4\}$ by

$$a_0/a_4 = -d_2c_3c_4, \quad (45)$$

$$a_1/a_4 = d_2c_4 + d_2c_3 + d_1c_3c_4, \quad (46)$$

$$a_2/a_4 = d_2 - d_1(c_4 + c_3) + c_3c_4, \quad (47)$$

$$a_3/a_4 = d_1 - c_3 - c_4. \quad (48)$$

7.) If two of the roots are conjugate complex and the other two are degenerate real ($c_4 = c_3$ in (45) - (48)), the related solution of (22) reads

$$-\frac{2d_1c_3 + d_1^2 + 2c_3^2 + 2d_2}{(d_2 - d_1c_3 - c_3^2)^2\sqrt{-4d_2 - d_1^2}}\arctan\frac{2\chi + d_1}{\sqrt{-4d_2 - d_1^2}} + \frac{(2c_3 + d_1)\ln(\chi - c_3)}{(d_2 - d_1c_3 - c_3^2)^2} - \frac{(2c_3 + d_1)\ln(\chi^2 + d_1\chi - d_2)}{2(d_2 - d_1c_3 - c_3^2)^2}$$
$$- \frac{1}{(d_2 - d_1c_3 - c_3^2)(\chi - c_3)} + a_4(\xi - \xi_0) = 0. \quad (49)$$

8.) If (33) possesses two sets of nondegenerate conjugate complex roots, the general solution of (22) becomes

$$\begin{aligned}&\frac{(2d_4+d_1d_3-d_1^2-2d_2)}{\sqrt{-4d_2-d_1^2}}\arctan\frac{2\chi+d_1}{\sqrt{-4d_2-d_1^2}}+\frac{1}{2}(d_3-d_1)\ln(\chi^2+d_1\chi-d_2)\\&+\frac{2d_2-2d_4-d_3^2+d_1d_3}{\sqrt{-4d_4-d_3^2}}\arctan\frac{2\chi+d_3}{\sqrt{-4d_4-d_3^2}}-\frac{1}{2}(d_3-d_1)\ln(\chi^2+d_3\chi-d_4)\\&+(d_4^2-2d_2d_4-d_3^2d_2+d_1d_3d_4+d_2^2-d_1^2d_4+d_1d_2d_3)a_4(\xi-\xi_0)=0,\end{aligned}\tag{50}$$

where

$$d_1^2+4d_2<0,\quad d_3^2+4d_4<0,\tag{51}$$

and a_i and d_i are related by

$$a_0/a_4=d_2d_4,\quad a_1/a_4=-d_2d_3-d_1d_4,\tag{52}$$

$$a_2/a_4=-d_2-d_4+d_3d_1,\quad a_3/a_4=d_1+d_3.\tag{53}$$

9.) Finally, if $d_1=d_3$ and $d_2=d_4$ in (52) and (53), the χ function should satisfy the relation

$$\frac{2\chi+d_1}{(\chi^2+d_1\chi-d_2)}+\frac{4}{\sqrt{-4d_2-d_1^2}}\arctan\frac{2\chi+d_1}{\sqrt{-4d_2-d_1^2}}+a_4(-4d_2-d_1^2)(\xi-\xi_0)=0.\tag{54}$$

If the condition (25) is not satisfied, we have five further possible cases:
(i) If the parameters v_2,v_3 and C_1 are taken as

$$C_1=0,\quad v_3=\frac{15v_4^2}{64v_5},\quad v_2=0,\tag{55}$$

then the χ function is given by

$$\frac{1}{\sqrt{10}(2b_2\chi+b_1)}-\frac{\sqrt{v_5}}{5\sqrt{-b_2v_4}}\operatorname{arctanh}\Big(\frac{\sqrt{2v_5}(2b_2\chi+b_1)}{\sqrt{-5b_2v_4}}\Big)+\frac{v_4}{32\sqrt{v_5b_2}}(\xi-\xi_0)=0,\tag{56}$$

and the corresponding solution of the ϕ^5 model with (55) reads

$$\phi=\frac{b_1^2}{4b_2}+b_1\chi+b_2\chi^2\tag{57}$$

with three arbitrary constants $b_1,\ b_2$ and ξ_0.
(ii) If the parameters v_2,v_3 and C_1 are given by

$$v_2=-\frac{9v_4^3}{32v_5^2},\quad v_3=-\frac{3v_4^2}{16v_5},\quad C_1=\frac{27v_4^5}{640v_5^4},\tag{58}$$

the related χ obeys the form

$$-\frac{1}{2b_2\chi+b_1}-\frac{\sqrt{v_5}}{\sqrt{5b_2v_4}}\operatorname{arctanh}\Big(-\frac{\sqrt{v_5}(2b_2\chi+b_1)}{\sqrt{5b_2v_4}}\Big)+\frac{\sqrt{10}v_4}{16\sqrt{v_5b_2}}(\xi-\xi_0)=0,\tag{59}$$

and ϕ has the form

$$\phi=\frac{-3v_4b_2+v_5b_1^2}{4v_5b_2}+b_1\chi+b_2\chi^2.\tag{60}$$

(iii) The constraints on the parameters for the third special solution read

$$C_1 = 0, \quad v_2 = \frac{25v_4^3}{864v_5^2}, \quad v_3 = \frac{5v_4^2}{16v_5}, \tag{61}$$

while the corresponding χ and ϕ are given by

$$-\frac{\sqrt{10}}{100(2b_2\chi + b_1)} - \frac{150v_5}{500\sqrt{b_2v_4}}\operatorname{arctanh}\Big(\frac{-\sqrt{15v_5}(2b_2\chi + b_1)}{\sqrt{b_2v_4}}\Big) = -\frac{v_4}{480\sqrt{v_5b_2}}(\xi - \xi_0) \tag{62}$$

and

$$\phi = \frac{3v_5b_1^2 - 5v_4b_2}{12v_5b_2} + b_1\chi + b_2\chi^2, \tag{63}$$

respectively.

(iv) For the fourth special solution we have constraints on the parameters v_2, v_3 and the first integration constant C_1:

$$v_3 = \frac{3(\sqrt{5}+1)v_4^2}{32v_5}, \quad v_2 = -\frac{v_4^3(29\sqrt{5} - 60)}{16v_5^2(7\sqrt{5} - 27)}, \quad C_1 = 0. \tag{64}$$

The related χ function is given by

$$\sqrt{\frac{1}{\sqrt{5}} + 3}\,\operatorname{arctanh}\Big(\frac{\sqrt{v_5(5+3\sqrt{5})}(2b_2\chi + b_1)}{\sqrt{10v_4b_2}}\Big) - \sqrt{2}\operatorname{arctanh}\Big(\frac{5\sqrt{v_5}(2b_2\chi + b_1)}{\sqrt{\sqrt{5}v_4b_2}}\Big)$$
$$-\frac{v_4^{3/2}(\sqrt{5} - 1)(\xi - \xi_0)}{5^{1/4}v_5} = 0, \tag{65}$$

and the solution of the ϕ equation is

$$\phi = \frac{v_5b_1^2 - \sqrt{5}v_4b_2}{4v_5b_2} + b_1\chi + b_2\chi^2. \tag{66}$$

(v) The final special solution has the form

$$\phi = \frac{v_5b_1^2 - 2v_4b_2 + \sqrt{5}v_4b_2}{4v_5b_2} + b_1\chi + b_2\chi^2, \tag{67}$$

where the function χ is related to ξ implicitly by

$$\sqrt{\frac{5+\sqrt{5}}{2}}\operatorname{arctanh}\Big(\frac{\sqrt{v_5(5+3\sqrt{5})}(2b_2\chi + b_1)}{\sqrt{-b_2v_4}}\Big) - 5^{1/4}\operatorname{arctanh}\Big(\frac{v_5(2b_2\chi + b_1)}{\sqrt{-\sqrt{5}b_2v_4}}\Big)$$
$$+\frac{\sqrt{10}(\sqrt{5} - 1)v_4\sqrt{-v_4}(\xi - \xi_0)}{32v_5} = 0, \tag{68}$$

while the constrained conditions for the model parameters v_2, v_3 and the integral constant C_1 are

$$v_3 = \frac{3(\sqrt{5}+1)v_4^2}{32v_5}, \quad v_2 = -\frac{v_4^3(29\sqrt{5} - 60)}{16v_5^2(7\sqrt{5} - 27)}, \quad C_1 = \frac{v_4^5(4525655302009\sqrt{5} - 10119671799191)}{1280v_5^4(204037668464\sqrt{5} - 456248189781)}. \tag{69}$$

The parameters b_1, b_2, and ξ_0 in the above special cases are all arbitrary constants.

4. Solutions of the ϕ^8 Model

If we take $N = 8,\ v_1 = v_3 = v_5 = v_7 = 0$ in (4), we get the ϕ^8 model

$$\Box\Phi = V_2\Phi + V_4\Phi^3 + V_6\Phi^5 + V_8\Phi^7, \tag{70}$$

where we have written $v_i, i = 2,4,6,8$ and ϕ as V_i and Φ for convenience later. By means of the basic equation approach proposed in Sect. 2, some types of special solutions of the ϕ^8 model (70) can be obtained by solving a similar ODE:

$$A\Phi_\psi + B\Phi_{\psi\psi} = V_2\Phi + V_4\Phi^3 + V_6\Phi^5 + V_8\Phi^7, \tag{71}$$

where the auxilialy field ψ and the arbitrary functions $A = A(\psi)$ and $B = B(\psi)$ are related by (6) and (7). Especially, if A and B satisfy (12), we have

$$\Phi_\xi^2 = V_2\Phi^2 + \frac{1}{2}V_4\Phi^4 + \frac{1}{3}V_6\Phi^6 + \frac{1}{4}V_8\Phi^8 - \frac{c}{12}\Big(24V_2 + 18V_4c + 16V_6c^2 + 15V_8c^3\Big), \tag{72}$$

where ξ is given in (13) and c is an arbitrary integration constant.

It is interesting and straightforword to see that the solutions of (72) can be obtained simply by using the transformation

$$\Phi = \sqrt{c+\phi}, \tag{73}$$

where ϕ is a solution of the ϕ^5 model (21) with

$$v_5 = \frac{5}{2}V_8, \tag{74}$$

$$v_4 = \frac{8}{3}V_6 + 10cV_8, \tag{75}$$

$$v_3 = 3V_4 + 16cV_6 + 15c^2V_8, \tag{76}$$

$$v_2 = 4V_2 + 6cV_4 + 8c^2V_6 + 10c^3V_8, \tag{77}$$

$$C_1 = 4c^2(V_2 + cV_4 + c^2V_6 + c^3V_8). \tag{78}$$

So, all the special solutions obtained in Sects. 2 and 3 can be transformed to those of the Φ^8 model.

5. Some Special Solutions of Coupled Scalar Fields

Actually, the special solutions of the ϕ^5 model obtained in sections 2 and 3 may be used to get exact solutions of other physically significant models, say the coupled nonlinear scalar field model

$$\Box f = A_1f + A_2f^3 + A_3fg^2, \tag{79}$$

$$\Box g = B_1g + B_2g^3 + B_3gf^2 \tag{80}$$

which appears in some physical fields such as particle physics and field theory [8] and condense matter physics [9].

After some complicated but direct calculations, we can change the solutions of the ϕ^5 model obtained in Sections 2 and 3 to those of the coupled scalar fields f and g for some special parameters A_i and B_i. Here are five possible examples.

Case 1

If the parametes B_1 and B_3 are related to other parameters by

$$B_1 = -\frac{A_1B_2}{11B_2 - 6A_3}, \quad B_3 = \frac{5}{9}A_2, \tag{81}$$

a special type of the coupled scalar fields reads

$$g = \phi, \tag{82}$$

$$f^2 = \frac{\pm 9}{50A_2\sqrt{A_1}}\Big(3(6A_3 - 11B_2)(A_3 - 5B_2) \cdot (15B_2 - 8A_3)\Big)^{1/2}\phi^3 + \frac{9}{25A_2}(15B_2 - 8A_3)\phi^2 - \frac{\pm 9}{25A_2}\sqrt{\frac{3A_1(5B_2 - A_3)(15B_2 - 8A_3)}{(11B_2 - 6A_3)}}\phi - \frac{27}{25}\frac{(15B_2 - 8A_3)A_1}{(11B_2 - 6A_3)A_2}, \tag{83}$$

where ϕ is an arbitrary solution of the ϕ^5 model given in Sect. 3, or more generally by (14) with

$$C_1 = \frac{24(15B_2 - 8A_3)A_1^2}{25(11B_2 - 6A_3)^2}, \tag{84}$$

$$v_3 = \mp\frac{\sqrt{3A_1(5B_2 - A_3)(15B_2 - 8A_3)}}{5\sqrt{(11B_2 - 6A_3)}},$$

$$v_4 = 4B_2 - \frac{8}{5}A_3,$$

$$v_2 = \frac{2A_1(12A_3 - 25B_2)}{5(11B_2 - 6A_3)}, \tag{85}$$

$$v_5 = \pm\frac{1}{10\sqrt{A_1}} \cdot \sqrt{3(11B_2 - 6A_3)(5B_2 - A_3)(15B_2 - 8A_3)}.$$

Case 2

If the parameters B_i $(i = 1, 2, 3)$ are related to A_i by

$$B_1 = -\frac{13}{9}A_1,\ B_2 = \frac{-64}{287955}A_3(5A_\pm - 1539),$$

$$B_3 = \frac{5}{9}A_2 \tag{86}$$

with

$$A_\pm = -\frac{11051}{128} \pm \frac{553}{\sqrt{128553}}, \tag{87}$$

and the parameters of the solutions of the ϕ^5 model given in Sects. 2 and 3 are replaced by

$$C_1 = 0,\ v_2 = -2A_1,\ v_3 = \frac{2}{9}B_\pm,$$

$$v_4 = \frac{-8}{287955}A_3(8343 + 160A_\pm), \tag{88}$$

$$v_5 = \frac{4}{518319B_\pm}A_3^2(832A_\pm + 124011)$$

with

$$B_\pm = \frac{1}{22110}\sqrt{(A_3A_1(229 \pm \sqrt{7553}))}, \tag{89}$$

then a special solution of (79) and (80) has the form

$$g = \phi, \tag{90}$$

$$f^2 = \frac{4A_3^2}{287955A_2}(832A_\pm + 124011)\phi^3 \tag{91}$$

$$- \frac{8A_3}{159975A_2}(20655+152A_\pm)\phi^2 + \frac{1}{5A_2}B_\pm\phi - \frac{A_1}{A_2}.$$

Case 3

When the parameters B_i are fixed by

$$B_1 = 4A_1,\ B_2 = \frac{14}{45}A_3,\ B_3 = \frac{5}{9}A_2, \tag{92}$$

we can obtain a special solution of (79) and (80) with the form

$$g = \phi, \tag{93}$$

$$f^2 = \mp\frac{\sqrt{-55A_1A_3^3}}{50A_1A_2}\phi^3 - \frac{6A_3}{5A_2}\phi^2 \pm \frac{18}{55A_2}\sqrt{-55A_3A_1}\phi, \tag{94}$$

where ϕ is given by (14) or ϕ in Sect. 3, while the constants v_i and C_1 should be replaced by

$$C_1 = 0,\ v_2 = 4A_1,\ v_3 = \pm\frac{2}{11}\sqrt{-55A_3A_1},$$

$$v_4 = -\frac{16}{45}A_3, v_5 = \mp\frac{1}{90A_1}A_3\sqrt{-55A_3A_1}. \tag{95}$$

Case 4

For

$$B_1 = \frac{1}{3}A_1,\ B_2 = \frac{1}{5}A_3,\ B_3 = \frac{5}{9}A_2, \tag{96}$$

a special type of solutions of the coupled scalar fields reads

$$g = \phi, \tag{97}$$

$$f^2 = \frac{9}{5A_2}F_5\phi^3 - \frac{9}{5A_2}A_3\phi^2 - \frac{A_1}{A_2}, \tag{98}$$

where ϕ is given by (14) or ϕ in Sect. 3, while the constants v_i and C_1 should be replaced by

$$C_1 = 0,\ v_2 = -\frac{2}{9}A_1,\ v_3 = 0,\ v_4 = -\frac{4}{5}A_3,$$

$$v_5 = \text{arbitrary constant}. \tag{99}$$

Case 5

The final one has the form

$$g = \phi, \tag{100}$$

$$\psi = \sqrt{\frac{9F_5}{5A_2}\phi^3}, \tag{101}$$

where the B_i are fixed as

$$B_1 = \frac{4}{9}A_1,\ B_2 = \frac{8}{15}A_3,\ B_3 = \frac{5}{9}A_2, \tag{102}$$

and v_i and C_1 of the ϕ^5 model should be replaced by

$$C_1 = 0,\ v_2 = \frac{4}{9}A_1,\ v_3 = 0,\ v_4 = \frac{8}{15}A_3,$$
$$v_5 = \text{arbitrary constant}. \qquad (103)$$

From the field equations (79) and (80) we know that, if we take the transformation

$$A_i \leftrightarrow B_i,\ f \leftrightarrow g \qquad (104)$$

for Case 1 to Case 5, we can get another five special types of solutions for the coupled scalar fields.

6. Summary

In this paper, using an arbitrary single scalar field equation with a constraint condition as the basic equation system, we obtain a special type of exact solutions of the ϕ^5 model by solving an ODE. For special types of the constraint, the ODE is solved by direct integration. For the solitary wave solutions we solve the problem by a nonstandard truncation approach of the extended Painlevé approach developed in [7]. Fourteen types of exact solutions of the ϕ^5 model are obtained from the nonstandard truncation approach.

It is interesting that the exact solutions of the ϕ^5 model may be mapped to the solutions of many other kinds of physically useful models. In this paper we have obtained some exact solutions of two physically significant models: (i) Making a simple transformation, all the exact solutions of the ϕ^5 model obtained from the basic equation approach can be changed to those of the ϕ^8 model. (ii) For the coupled nonlinear scalar field model which is used in field theory and condensed matter physics, there may be abundant solitary wave structures [10]. For some special parameters of the coupled scalar field equations we obtained ten types of special solutions by means of the ϕ^5 model.

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